Real Permanental Roots of Doubly Stochastic Matrices

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ABSTRACT

It is shown that for each $n \geqslant 7$ there exists an $n \times n$, irreducible, doubly stochastic matrix A such that the permanent of the characteristic matrix of A has n real zeros.

The *permanental roots* of a square matrix A are the zeros of the permanent of the characteristic matrix of A. A nonnegative real matrix with all row and column sums equal to 1 is called doubly stochastic. It was conjectured by de Oliveira [3] that an irreducible doubly stochastic matrix can have at most one real permanental root. Hartfiel [4] presented counterexamples of orders four and five for this conjecture. More generally, Csima [2] showed that for each positive integer k there exists a positive doubly stochastic matrix A of order $3k$ such that A has at least k distinct real permanental roots. Levow [5] later proved that for each integer $n \geqslant 3$ there is an $n \times n$ irreducible doubly stochastic matrix with $n-2$ distinct real permanental roots.

Levow proposed that his result was the best possible in the sense that no doubly stochastic matrix other than the identity has only real permanental roots. It is not difficult to show that this is true for matrices of order no greater than three. However, it is false for large matrices.

THEOREM 1. *For each integer $n \geqslant 7$ there exists an $n \times n$ irreducible doubly stochastic matrix with n real permanental roots.*

Proof. Let $n \geqslant 7$, and define an $n \times n$ matrix $A = (a_{ij})$ by letting all $a_{ij} = 0$ except

$$a_{kk} = \frac{n-2}{n-1}, \qquad a_{nk} = a_{kn} = \frac{1}{n-1}, \qquad k = 1, \ldots, n-1.$$

0024-3795/78/0021-0289$1.25

Then A is an $n\times n$ irreducible doubly stochastic matrix. Let

$$B=(n-1)A+(2-n)I.$$

Observe that λ is a permanental root of A if and only if $(n-1)\lambda+2-n$ is a permanental root of $B=(b_{ij})$. All $b_{ij}=0$ except

$$b_{nn}=2-n, \qquad b_{nk}=b_{kn}=1, \qquad k=1,\dots,n-1.$$

Hence,

$$\operatorname{per}(\lambda I-B)=\lambda^n+(n-2)\lambda^{n-1}+(n-1)\lambda^{n-2}=\lambda^{n-2}\left[\lambda^2+(n-2)\lambda+n-1\right].$$

Since $n\geqslant 7$, we see that $(n-2)^2>4(n-1)$, and it follows that B has n real permanental roots. Therefore A has n real permanental roots. ∎

We now show that for $n\geqslant 7$ there is no restriction on how many of the permanental roots of an $n\times n$ irreducible doubly stochastic matrix A can be real other than the obvious restriction imposed by $\operatorname{per}(\lambda I-A)$ being a real polynomial. Denote the number of real permanental roots of A by $N(A)$.

THEOREM 2. *Let n and t be integers with $n\geqslant 7$ and $0\leqslant t\leqslant n/2$. Then there exists an $n\times n$ irreducible doubly stochastic matrix A such that $N(A)=n-2t$.*

Proof. If $t=0$ the conclusion follows from Theorem 1. We complete the proof of this theorem by proving the following.

LEMMA. *Let n and t be integers with $n\geqslant 2$ and $1\leqslant t\leqslant n/2$. Then there exists an $n\times n$ positive doubly stochastic matrix A such that $N(A)=n-2t$ and the real permanental roots of A are distinct.*

Proof. We induct on n. It is easy to see that the lemma holds for $n=2$. Let $n>2$. If $t=n/2$, then n is even, and it follows that the $n\times n$ matrix J_n with each entry equal to $1/n$ is a positive doubly stochastic matrix with $N(J_n)=0$. Suppose that $t<n/2$. Then $t\leqslant(n-1)/2$, and it follows from the inductive assumption that there exists an $(n-1)\times(n-1)$ positive doubly stochastic matrix A such that $N(A)=n-1-2t$, where the $N(A)$ real permanental roots of A are distinct. Since A is positive, we see that $\lambda<1$ for each real permanental root λ of A, from the discussion by Brualdi and Newman [1, p. 238] of the circumstances under which equality holds in the inequality $\operatorname{per}(I-A) \geqslant 0$ for a stochastic matrix A. Hence $1 \oplus A$ is an $n \times n$ doubly stochastic matrix such that $N(1 \oplus A) = 1 + N(A) = n - 2t$ and the real permanental roots of $1 \oplus A$ are distinct. It now follows from a continuity argument that for $\varepsilon > 0$ sufficiently small the $n \times n$ matrix

$$B = \varepsilon J_n + (1-\varepsilon)(1 \oplus A)$$

is an $n \times n$ positive doubly stochastic matrix such that $N(B) = n - 2t$ and the real permanental roots of B are distinct. ∎

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Received 22 February 1977